18.06 Professor Edelman Quiz 2 November 7, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Andrey Grinshpun	2-349	3-7578	agrinshp
2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	agrinshp
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	ghorel
6	T 1	2-132	Tiankai Liu	2-491	3-4091	tiankai
7	T 2	2-132	Tiankai Liu	2-491	3-4091	tiankai

1 (27 pts.)

P is any $n \times n$ Projection Matrix. Compute the ranks of A,B, and C below. Your method must be visibly correct for every such P, not just one example.

a) (8 pts.) $A = (I - P)P$.

Since P is a projection matrix, $P^2 = P$, so $(I-P)P = P - P^2 = P - P = 0$ and has rank 0.

b) (10 pts.) $B = (I - P) - P$. (Hint: Squaring B might be helpful.)

$B^2 = (I - 2P)^2 = I^2 - 4P + 4P^2 = I$. The rank of I is n. The rank of B^2 is at most the rank of B and the rank of B is at most n, so B must have rank n.

c) (9 pts.) $C = (I - P)^{2012} + P^{2012}$.

Note $I - P$ is a projection matrix, so $(I - P)^{2012} = I - P$ and $P^{2012} = P$, so the above simplifies to I, which has rank n.

2 (22 pts.)

Consider a 4×4 matrix

$$A = \begin{pmatrix} 0 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}.$$

a) (17 pts.) Compute $|A|$, the determinant of A, in simplest form.

The answer is $\det A = -x^2 - y^2 - z^2$. But before we discuss how to get this answer, I'd like to call your attention to that fact that the expression $-x^2 - y^2 - z^2$ is symmetric in the three variables x, y, z. That is to say, if we swap the roles of any two of these variables, the expression as a whole is unchanged. Why might we have predicted that $\det A$ has this property? Well, if we swap rows 2 and 3 of A, and then swap *columns* 2 and 3 of the result, we end up with

$$A' = \begin{pmatrix} 0 & y & x & z \\ y & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix},$$

which is the same as A, but with the roles of x and y swapped. In performing one row swap and one column swap, we have multiplied the determinant by $(-1)^2 = 1$, so A' has the same determinant as A. From this we conclude that $\det A$, whatever it is, must be an expression that's symmetric in x and y. Similar considerations show that it's symmetric in all three variables x, y, z.

Anyway, let's actually compute $\det A$. Here were some of the most common ways from the students' tests:

- By cofactor expansion (p. 260) in the first row (or the first column), using the big formula (p. 257) or any other method for each 3×3 minor:

$$\begin{aligned}\det A &= 0\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - x\begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} + y\begin{vmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix} - z\begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix} \\ &= 0 - x(x) + y(-y) - z(z) \\ &= -x^2 - y^2 - z^2.\end{aligned}$$

  Note the alternating + and − signs in the cofactors:

$$C_{11} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \ C_{12} = -\begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix}, \ C_{13} = \begin{vmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix}, \ C_{14} = -\begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix}.$$

  In general, the formula is $C_{ij} = (-1)^{i+j} \det M_{ij}$.

- By cofactor expansion in the *second* row (or the second column), using the big formula or any other method for each 3×3 minor:

$$\begin{aligned}\det A &= -x\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & y & z \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & x & z \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & x & y \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix} \\ &= -x(1) + 1(-y^2 - z^2) \\ &= -x^2 - y^2 - z^2.\end{aligned}$$

  The cofactor C_{22}, for example, can be calculated using the big formula for $3{\times}3$ matrices:

$$\begin{vmatrix} 0 & y & z \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} = 0 \cdot 1 \cdot 1 + y \cdot 0 \cdot z + z \cdot y \cdot 0 - 0 \cdot 0 \cdot 0 - y \cdot y \cdot 1 - z \cdot 1 \cdot z = -y^2 - z^2.$$

- By the big formula (pp. 258–259) for 4×4 matrices. The big formula has 24 terms (one for each 4×4 permutation matrix), but only three of them are nonzero:

$$\det A = x^2 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + y^2 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + z^2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

These three permutation matrices all have determinant -1, because they are one row exchange away from the identity matrix, so

$$\det A = -x^2 - y^2 - z^2.$$

- By performing row operations to reach an upper triangular matrix. First exchange row 1 with another row to put a pivot in the top-left corner; to make the future computations simpler, let's swap row 1 with row 4:

$$|A| = - \begin{vmatrix} z & 0 & 0 & 1 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & x & y & z \end{vmatrix};$$

here we have a $-$ sign because row exchanges negate the determinant (rule 2, p. 246). Now subtract x/z times row 1 from row 2, and y/z times row 1 from row 3:

$$|A| = - \begin{vmatrix} z & 0 & 0 & 1 \\ 0 & 1 & 0 & -x/z \\ 0 & 0 & 1 & -y/z \\ 0 & x & y & z \end{vmatrix};$$

remember that such operations do not affect the determinant (rule 5, p. 247). Finally subtract x times row 2 and y times row 3 from row 4:

$$|A| = -\begin{vmatrix} z & 0 & 0 & 1 \\ 0 & 1 & 0 & -x/z \\ 0 & 0 & 1 & -y/z \\ 0 & 0 & 0 & z + x^2/z + y^2/z \end{vmatrix}.$$

Now we can mutiply the diagonal entries (rule 7, p. 247) to find that

$$|A| = -z \cdot 1 \cdot 1 \cdot (z + x^2/z + y^2/z) = -x^2 - y^2 - z^2.$$

- By performing row operations to reach a *lower* triangular matrix. From row 1 of A, we subtract x times row 2, y times row 3, and z times row 4. These operations do not change the determinant, so

$$|A| = \begin{vmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = -x^2 - y^2 - z^2.$$

In other words, we may factorize A as

$$A = \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix},$$

so the product rule (rule 9, p. 248) says

$$|A| = \begin{vmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = -x^2 - y^2 - z^2.$$

b) (5 pts.) For what values of x, y, z is A singular?

A square matrix is singular if and only if its determinant equals zero. So we are asked to find all triples (x, y, z) such that

$$\det A = -x^2 - y^2 - z^2 = 0,$$

or in other words

$$x^2 + y^2 + z^2 = 0.$$

So far, we have been talking about real numbers x, y, z in this course, so the left-hand side is just the square of the distance from (x, y, z) to the origin in $\mathbb{R}^3$. Since only the origin is at a distance 0 from the origin, the matrix A is singular if and only if $x = y = z = 0$.

3 (22 pts.)

The 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has QR decomposition

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = Q \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

a) (7 pts.) What is r_{11} in terms of the variables $a, b, c, d, e, f, g, h, i$? (but not any of the elements of Q.)

You should probably remember that r_{11} is the norm of the first column of the matrix on the left, which we will call A. But let's rederive it. So, when we do a QR decomposition, we always start with the first column of our matrix, here the vector $(a\ d\ g)^T$, and we normalize it to obtain the first column of Q: $q_1 = (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\|$. Now, if we look at the first column of $A = QR$, we have $(a\ d\ g)^T = r_{11} \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3 = r_{11} \cdot q_1 = r_{11} \cdot (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\|$, which implies that $r_{11} = \left\|(a\ d\ g)^T\right\| = \sqrt{a^2 + d^2 + g^2}$.

b) (15 pts.) Solve for x in the equation,

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

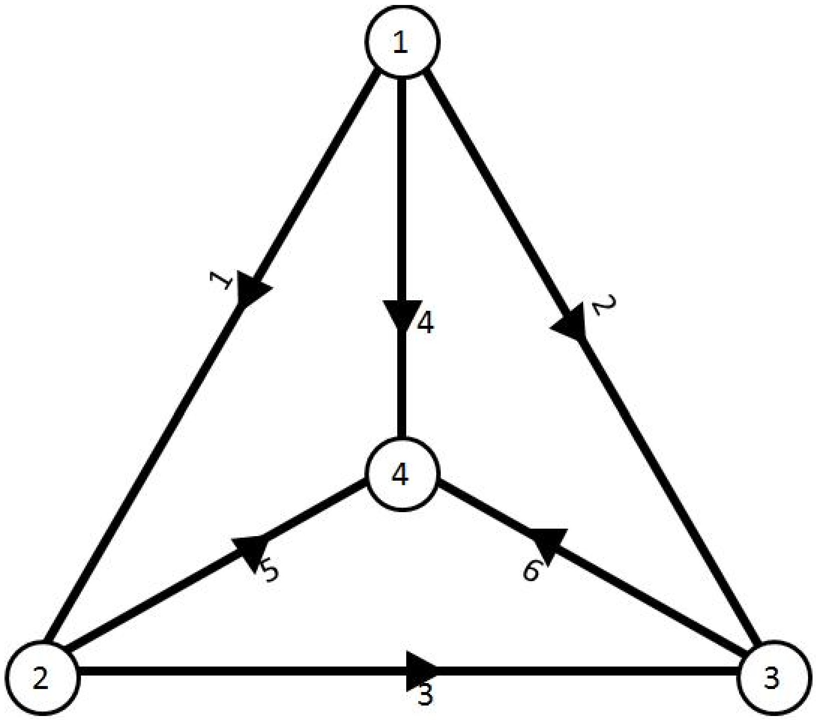
expressing your answer possibly in terms of r_{11}, r_{22}, r_{33} and the variables $a, b, c, d, e, f, g, h, i$, (but not any of the elements of Q.)

Look at the product

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

row by row: we take the first row of Q^T, that is, the first column of Q, and take its dot product with x to obtain 1. We also take the second and third row of Q^T, that is, the second and third column of Q, and take their dot products with x to obtain 0. This means that x is perpendicular to the last 2 columns of Q. But because Q has orthonormal columns and we are in R^3, this can only mean that x is a multiple of the first column, say $x = zq_1$ for some real number z. But remember we said that $q_1^T x = 1$, which means $q_1^T zq_1 = zq_1^T q_1 = 1$, but we know $q_1^T q_1 = 1$ because the columns of Q have norm 1. So clearly $z = 1$ and x is q_1, which we found in the previous question. So $x = q_1 = (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\| = (a\ d\ g)^T / r_{11}$.

4 (29 pts.)



a) (15 pts.) Use loops or otherwise to find a basis for the left nullspace of the incidence matrix A for the graph above. We will start you off, one basis vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

The incidence matrix is 6 by 4. Since the graph is connected, the nullspace has dimension 1, it is the line generated by $(1, 1, 1, 1)^T$, therefore, the matrix has rank 3. It follows that the left nullspace has dimension $6 - 3 = 3$.

We use the result of page 425 of the book :

A basis of the left nullspace of the incidence matrix is given by a set of independant loops. In this case, we need to find 3 independant loops in the graph. It is easy to check that the 3 small loops are independant :

A basis of the left nullspace is :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

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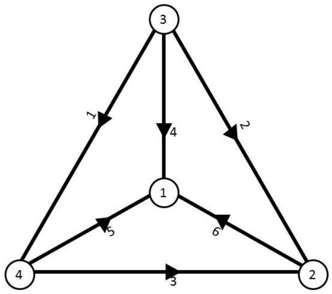

There are 24 ways to relabel the four nodes in the graph in part(a). Edge labels remain unchanged. One of the 24 ways is pictured above. This produces 24 incidence matrices A.

b) (7 pts.) Is the row space of A independent of the labelling? Argue convincingly either way.

Yes it is independent. Indeed, the incidence matrix of a connected graph with 4 nodes has the line generated by $(1,1,1,1)^T$ as its nullspace whatever the graph is. In particular, we see that the nullspace is independant of the labelling. Since the row space is the orthogonal of the nullspace it is also independant of the labelling.

c) (7 pts.) Is the column space of A independent of the labelling? Argue convincingly either way.

Yes, it is independent. Relabelling the nodes has the effect of permuting the columns of the incidence matrix. The columns space is the space of linear combinations of the columns of the matrix, therefore, it is independent of the way the columns are ordered in the matrix.

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